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One-dimensional discrete Stark Hamiltonian and resonance scattering by an impurity

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Abstract. A one-dimensional discrete Stark Hamiltonian with a continuous electric field is constructed by extension theory methods. In the absence of an impurity the model is proved to be exactly solvable, the spectrum is shown to be simple and continuous, filling the real axis; the eigenfunctions, the resolvent and the spectral measure are constructed explicitly. For this (unperturbed) system the resonance spectrum is shown to be empty.

The model considering an impurity in a single node is also constructed using the operator extension theory methods. The spectral analysis is performed and the dispersion equation for the resolvent singularities is obtained. The resonance spectrum is shown to contain an infinite discrete set of resonances. One-to-one correspondence of the constructed Hamiltonian to some Lee–Friedrichs model is established.

1. Introduction

A one-dimensional Stark-type Hamiltonian on a line has been studied by many authors [1–12]. Most attention has been paid to the resonance structure of this system. The key ingredient of the models studied in [1–12] is the absolutely continuous spectrum filling the whole real axis. It provides the possibility of applying the powerful methods of scattering theory to the study of the spectrum of resonances.

However, it is well known [13, 14] that a discrete one-dimensional Stark Hamiltonian on a lattice has a discrete spectrum only. This means that there are no 'scattering states' even for an unperturbed system. This prevents the study of the spectrum of resonances caused by the perturbation of an ideal crystal lattice.

In the present paper we study a model for the motion of an electron on a one-dimensional lattice in a homogeneous electric field and electron resonance scattering by an impurity treated as a perturbation. The main idea of our approach is to put the dynamical variables and equation of motion on a spatial lattice, whereas the absolutely continuous spectrum is kept intact.

In order to construct a discrete Stark Hamiltonian with an absolutely continuous spectrum we treat the kinetic energy of an electron in a lattice as the operator of the

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second difference, whereas the electric field is considered as continuous, filling the intersite intervals. As is shown in the present paper, the discrete Stark operator with continuous electric field proposed here has an absolutely continuous spectrum in contrast to the discrete Stark operator with electric field located at the sites of the lattice [13, 14]. In our opinion the discrete Stark operator with continuous electric field seems to be more natural from the physical point of view, at least because the concept of a field needs continuity by itself. On the other hand, the suggested model seems also to be more sound from the spectral point of view because of the presence of a continuous spectrum, and as a consequence the presence of the propagating electronic waves in the system.

Treating such a Hamiltonian as an unperturbed operator, we construct the perturbed Hamiltonian which describes the interaction of the Stark electron with an impurity.

In the present paper we consider the single impurity localized at the site with the number $n = 0$. We ‘switch on’ the interaction between the Stark electron and the impurity by the extension theory methods [17, 18]. In contrast to the ordinary delta-like interaction our approach allows one to take into account the impurity internal degrees of freedom. The advantage is that the perturbed Stark operator describing the resonance scattering by impurity leads to the exactly solvable model having at the same time a reach set of resonances. We calculate the location of resonances by perturbation theory methods in a weak coupling limit.

Let us emphasize that this paper does not consider the Wannier–Stark problem, because in our model the kinetic energy operator is a difference operator.

We reduce the proposed model to some Lee–Friedrichs model [15, 16] and treat the latter as a particular case of models based on the extension theory [17, 18]. This reduction can be useful when studying the transition to chaos and the problem of intrinsic irreversibility [19] for discrete Stark Hamiltonians.

The paper is organized as follows. In section 2 we construct the discrete Stark Hamiltonian with continuous electric field and perform its spectral analysis explicitly. In section 3, by means of the extension theory methods, we construct the perturbed Hamiltonian describing the interaction of the Stark electron with an impurity. In the same section we make the analytic continuation of the resolvent bilinear form and in terms of this continuation calculate the spectrum of resonances. In section 4 we reduce our model to the Friedrichs–Lee model and discuss on this basis the applicability of the generalized spectral decompositions, in connection with the intrinsic irreversibility problem and chaotic regimes.

2. Unperturbed Hamiltonian

In this section we construct a Hamiltonian describing a chain of sites embedded in a continuous electric field and study its spectral properties.

Let us consider the discrete Stark operator H_d acting in the Hilbert space $\mathcal{H}_d = l^2$,

$$(H_d\Psi)_n = -\frac{1}{(2\pi a)^2}(\Psi_{n-1} + \Psi_{n+1}) + 2\pi\varepsilon an\Psi_n \tag{1}$$

and the multiplication operator

$$H_c = \varepsilon ay \qquad y \in (-\pi, \pi) \tag{2}$$

acting in the Hilbert space $\mathcal{H}_c = L^2(-\pi a, \pi a)$. Here $2\pi a$ is the intersite distance and $\varepsilon > 0$ is the electric field parameter.

We introduce the Hamiltonian describing a chain of sites in an electric field in the form

$$H = H_d \times I_c + I_d \times H_c. \tag{3}$$

Here H_d and H_c are given by equations (1) and (2), respectively, I_d and I_c are the identity operators in $\mathcal{H}_d$ and $\mathcal{H}_c$. The notation $\times$ stands for the operator tensor product. The operator H acts in the space

$$\mathcal{H} = l^2(\mathbb{Z}; L^2[-\pi, \pi)). \tag{4}$$

One can note that the potential

$$U = 2\pi a\varepsilon n \times I_c + I_d \times \varepsilon a y = \varepsilon a(2\pi n + y)$$

is a continuous electric field potential on a line. Indeed, any point $\nu \in \mathbb{R}$ can be parametrized by the site number $n \in \mathbb{Z}$ and the point of the interval $y \in [-\pi, \pi) : \nu = a(2\pi n + y)$. So in contrast to the discrete Stark Hamiltonian H_d, our model describes a chain of sites in a continuous electric field.

In what follows we call the operator H the unperturbed operator. The perturbation will be introduced in the next section as the impurity located in one of the sites.

In order to describe spectral properties of the unperturbed operator H we need the following notation. By angular brackets, $\langle *, * \rangle_{\mathcal{A}}$, we denote the inner product in a Hilbert space $\mathcal{A}$ and by square brackets, $[\alpha]$, the integer part of a real number α. The integer-valued function $M(\lambda)$ is defined as

$$M(\lambda) \stackrel{\text{def}}{=} \left[\frac{\lambda}{2\pi a\varepsilon} + \frac{1}{2}\right] \qquad \lambda \in \mathbb{R}.$$

$J_\nu(z)$ are the Bessel functions of first type. The components of the vector $\widehat{J}^{(m)} \in l^2$ are defined as follows: $(\widehat{J}^{(m)})_n = J_{n-m}(\Theta)$, where $\Theta = -(4\pi^3 a^3 \varepsilon)^{-1}$. The Heaviside step function is denoted by $\theta(x)$.

Lemma 1. The spectrum $\sigma(H)$ of the operator H is simple, absolutely continuous and fills the real axis $\mathbb{R}$. The wavefunctions are distributions and are given by

$$\Psi_n(y, \lambda) = J_{n-M(\lambda)}(\Theta)\delta(\varepsilon a y - \lambda + 2\pi M(\lambda)\varepsilon a). \tag{5}$$

The spectral family (resolution of the identity) of the operator H are projections in the space $\mathcal{H}$ of the form

$$E_\lambda(y) = \sum_{m=-\infty}^{\infty} \langle *, \widehat{J}^{(m)} \rangle_{l^2} \widehat{J}^{(m)} \theta(\lambda - 2\pi m\varepsilon a - \varepsilon a y). \tag{6}$$

Proof. The structure of the operator H leads to the separation of variables and therefore the spectral analysis of H is reduced to the spectral analysis of the operators H_d and H_c.

Let us use the Fourier transform in the space l^2, $F : l^2 \to L^2(-\pi, \pi)$:

$$(Ff)(q) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}nq} f_n.$$

In the Fourier representation the operator H_d turns into the operator

$$F H_d F^{-1} = -2\pi a\varepsilon \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}q} - \frac{2}{(2\pi a)^2}\cos q$$

acting in $L^2(-\pi, \pi)$ whose domain in the Sobolev space $H_2^1[-\pi, \pi)$ is determined by the periodicity condition $(Ff)(-\pi) = (Ff)(\pi)$. Solving the eigenvalue problem

$$(F H_d F^{-1} - \lambda)(Ff)(q) = 0$$

one can see that the spectrum of the operator FH_dF^{-1} is discrete and the eigenvalues are

$$\lambda_m = 2\pi a\varepsilon m \qquad m = 0, \pm 1, \pm 2, \ldots$$

whereas the correspondent eigenfunctions are

$$(Ff)^{(m)}(q) = \exp\{\mathrm{i}\Theta \sin q + \mathrm{i}mq\} \qquad \Theta = -(4\pi^3 a^3 \varepsilon)^{-1}.$$

Using the inverse Fourier transform one obtains

$$f_n^{(m)} = (2\pi)^{-1} \int_{-\pi}^{\pi} (Ff)^{(m)}(q)\, \mathrm{e}^{-\mathrm{i}nq}\, \mathrm{d}q = J_{n-m}(\Theta).$$

Here we have used the integral representation [29] of the Bessel function $J_\nu(\Theta)$.

Now consider the operator H_c in the Hilbert space $L^2(-\pi, \pi)$. Its spectrum is absolutely continuous and fills the interval $\zeta \in [-\pi\varepsilon a, \pi\varepsilon a]$. The correspondent continuous spectrum wavefunctions are distributions and have the form of delta functions $\psi_\zeta(y) = \delta(y - (\varepsilon a)^{-1}\zeta)$.

Due to the separation of variables the spectrum of the operator H is the algebraic sum of the spectra of the operators H_d and H_c:

$$\sigma(h) = \{z = z_1 + z_2 : z_1 \in \sigma(H_d), z_2 \in \sigma(H_c)\}.$$

Let us note that the length of the single spectral band of the operator H_c exactly coincides with the distance between the neighbouring eigenvalues of the operator H_d. Thus the spectrum of the operator H fills the real axis.

Any point $\lambda \in \mathbb{R}$ from the spectrum of H can be represented in the form

$$\lambda = \lambda_m + \nu \qquad m \in \mathbb{Z}, \nu \in [-\pi a\varepsilon, \pi a\varepsilon).$$

This representation corresponds to the energy distribution between the 'lattice' and 'field' subsystems determined by the operators H_d and H_c, respectively. As $\lambda_m = 2\pi a\varepsilon m$, $m = 0, \pm 1, \pm 2, \ldots,$ the 'mode number' m for a given energy λ is calculated as an integer-valued function

$$m = M(\lambda) = \left[\frac{\lambda}{2\pi a\varepsilon} + \frac{1}{2}\right].$$

Due to the separation of variables, the wavefunctions $\Psi_n(y)$ of the operator H are products of the corresponding eigenfunctions of the operators H_d and H_c and hence have the form (5).

Now we construct the spectral family $E_\lambda(y)$ of the operator H and show that the quadratic form

$$\eta(\lambda) = \langle E_\lambda \Phi, \Phi\rangle_{\mathcal{H}} \tag{7}$$

is an absolutely continuous function of λ for any $\Phi \in \mathcal{H}$. First, let us show that the resolvent $R(z) = (H - z)^{-1}$ is an operator-valued matrix $R(z) = \{R_{nn'}(z)\}$ with the entries

$$R_{nn'}(y, z) = \sum_{m=-\infty}^{\infty} \frac{J_{n-m}(\Theta) J_{n'-m}(\Theta)}{\lambda_m + \varepsilon a y - z} \tag{8}$$

which acts as a multiplication operator with respect to the variable y. Indeed, the resolvent $R_d(z) = (H_d - z)^{-1}$ of the operator H_d in the space l^2 obviously has matrix elements

$$(R_d(z))_{nn'} = \sum_{m=-\infty}^{\infty} \frac{J_{n-m}(\Theta) J_{n'-m}(\Theta)}{\lambda_m - z} \tag{9}$$

where $\lambda_m = 2\pi m\varepsilon a$ are the eigenvalues of the operator H_d. The resolvent $R_c(z) = (H_c - z)^{-1}$ of the operator H_c in the space $L^2(-\pi, \pi)$ is a multiplication operator

$$R_c(z)* = \frac{1}{\varepsilon a y - z} * . \tag{10}$$

Due to separation of variables the resolvent $R(z) = (H - zI)^{-1}$ can be calculated as a contour integral

$$\begin{aligned} R(z) &= (2\pi \mathrm{i})^{-1} \oint_\gamma R_c(\zeta) R_d(z-\zeta)\,\mathrm{d}\zeta \\ &= (2\pi \mathrm{i})^{-1} \int_{-\varepsilon a\pi}^{\varepsilon a\pi} R_d(z-\zeta)[R_c(\zeta + \mathrm{i}0) - R_c(\zeta - \mathrm{i}0)]\,\mathrm{d}\zeta \end{aligned}$$

where the contour γ encircles the spectrum of the operator H_c. On use of equations (9) and (10) the straightforward calculations give

$$\begin{aligned} R_{nn'}(y,z) &= (2\pi \mathrm{i})^{-1} \int_{-\varepsilon a\pi}^{\varepsilon a\pi} \sum_{m=-\infty}^{\infty} \frac{J_{n-m}(\Theta) J_{n'-m}(\Theta)}{\lambda_m - z + \zeta} \left(\frac{1}{\varepsilon a y - \zeta - \mathrm{i}0} - \frac{1}{\varepsilon a y - \zeta + \mathrm{i}0} \right) \mathrm{d}\zeta \\ &= \sum_{m=-\infty}^{\infty} J_{n-m}(\Theta) J_{n'-m}(\Theta) \int_{-\varepsilon a\pi}^{\varepsilon a\pi} \frac{\delta(\varepsilon a y - \zeta)}{\lambda_m - z + \zeta} \mathrm{d}\zeta = \sum_{m=-\infty}^{\infty} \frac{J_{n-m}(\Theta) J_{n'-m}(\Theta)}{\lambda_m - z + \varepsilon a y} \end{aligned}$$

which coincides with equation (8).

Resolution of the identity E_λ of any self-adjoint operator is related to the resolvent as follows [20]. If $(\alpha, \beta) \subset \mathbb{R}$ is an open interval, then in the strong operator topology

$$E_{(\alpha,\beta)} = \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} \int_{\alpha+\delta}^{\beta-\delta} (R(\lambda + \mathrm{i}\epsilon) - R(\lambda - \mathrm{i}\epsilon))\,\mathrm{d}\lambda.$$

Applying this formula to the resolvent $R(z)$ given by equation (8) one obtains resolution of the identity of the operator H in the form (6).

By means of equation (6) the function $\eta(\lambda)$ given by equation (7) takes the form

$$\eta(\lambda) = \sum_{p=-\infty}^{\infty} \int_{-\pi}^{\pi} \langle \Phi, \widehat{J}^{(p)} \rangle_{l^2} \langle \widehat{J}^{(p)}, \Phi \rangle_{l^2}\, \theta(\lambda - \lambda_p - \varepsilon a y)\,\mathrm{d}y.$$

It is clear that if $\lambda \in [\lambda_m - \pi\varepsilon a, \lambda_m + \pi\varepsilon a]$, $m \in \mathbb{Z}$, then

$$\eta(\lambda) = \sum_{p=-\infty}^{m-1} \int_{-\pi}^{\pi} \langle \Phi, \widehat{J}^{(p)} \rangle_{l^2} \langle \widehat{J}^{(p)}, \Phi \rangle_{l^2}\,\mathrm{d}y + \int_{-\pi}^{(\lambda-\lambda_m)/\varepsilon a} \langle \Phi, \widehat{J}^{(m)} \rangle_{l^2} \langle \widehat{J}^{(m)}, \Phi \rangle_{l^2}\,\mathrm{d}y.$$

Consequently, $\eta(\lambda_m + \pi\varepsilon a - 0) = \eta(\lambda_{m+1} - \pi\varepsilon a + 0)$. Hence the function $\eta(\lambda)$ and, consequently, the spectrum of the operator H, is absolutely continuous.

The lemma is proved. □

3. Perturbed Hamiltonian

In this section we assume that the chain of sites considered embedded in the electric field has an impurity. Namely we suppose that the electron dynamics governed by the Hamiltonian H is perturbed by the additional interaction between the electron and an impurity located at the single site with the number $n = 0$. We construct this interaction by means of extension theory methods [17, 18]. Namely we suppose that the impurity has an internal structure.

The dynamics of this internal structure is given by a self-adjoint operator H_i acting in an auxiliary Hilbert space $\mathcal{H}_i$. Then following the extension theory ideology we consider the extended space

$$\widehat{\mathcal{H}} = l^2(\mathbb{Z}, L^2(-\pi,\pi) \oplus \mathcal{H}_i) = \mathcal{H} \oplus l^2(\mathbb{Z}, \mathcal{H}_i)$$

as the state space for the system with the interaction.

Here we make the simplest choice $\mathcal{H}_i = \mathbb{C}$. Then

$$\widehat{\mathcal{H}} = \mathcal{H} \oplus l^2$$

and the self-adjoint operator H_i acting in the auxiliary space $\mathcal{H}_i$ is just the operator of multiplication by a real number $\mu \in \mathbb{R}$,

$$H_i \stackrel{\text{def}}{=} \mu \, * \, .$$

Let us embed the operator H in the space $\widehat{\mathcal{H}}$ as follows,

$$H \to \widehat{H} = H_d \times (I_c \oplus I_i) + I_d \times (H_c \oplus H_i) = (H_d \times I_c + I_d \times H_c) \oplus (H_d \times I_i + I_d \times H_i) = \begin{pmatrix} H & 0 \\ 0 & H_d \times I_i + I_d \times H_i \end{pmatrix} \tag{11}$$

where I_i stands for the identity operator in $\mathbb{C}$. The diagonal structure of this operator means that the embedding does not lead to any interaction between the Stark electron and the impurity.

In order to 'switch on' the interaction one can add to the diagonal operator-valued matrix (11) an off-diagonal self-adjoint operator V:

$$\widehat{H}_B = \widehat{H} + V \qquad V = \begin{pmatrix} 0 & B \\ B^+ & 0 \end{pmatrix} \tag{12}$$

where B is a bounded operator acting from l^2 to $\mathcal{H}$ and B^+ is its adjoint. Obviously $\widehat{H}_B$ is self-adjoint operator with the domain of $\widehat{H}$.

As we wish to have an additional interaction only with a single site (say, with $n = 0$), the operator B should vanish in the orthogonal complement to the linear span $\mathcal{L}\{\chi\}$ of the vector $\chi = (\ldots, 0, 0, 1, 0, 0, \ldots)^{\mathrm{T}} \in l_2$, $(\chi_n = \delta_{0n})$, i.e.

$$B|_{l^2 \ominus \mathcal{L}\{\chi\}} = 0.$$

This gives us the form† of the operator B:

$$B : f \mapsto \beta \langle f, \chi \rangle_{l^2} \widehat{\chi}. \tag{13}$$

Here $\beta \in \mathbb{R}$ is a coupling constant and $\widehat{\chi} = \chi \cdot \varphi(y) \in \mathcal{H}$. We consider here the interaction which does not depend on the field variable y, so it is reasonable to suppose that the function $\varphi(y) \in L^2[-\pi, \pi)$ is a constant, $\varphi(y) \equiv 1$, and

$$\widehat{\chi} = \chi \cdot \mathbf{1}(y) \in \mathcal{H}.$$

The adjoint operator acts as

$$B^+ : \Psi \mapsto \beta \langle \Psi, \widehat{\chi} \rangle_{\mathcal{H}} \chi. \tag{14}$$

To study the spectral properties of the operator $\widehat{H}_B$ we consider the spectral problem

$$\widehat{H}_B \begin{pmatrix} \Psi(y) \\ f \end{pmatrix} = z \begin{pmatrix} \Psi(y) \\ f \end{pmatrix} \qquad \Psi \in \mathcal{H}, \ f \in l_2 \tag{15}$$

† One can consider impurities localized in any finite number N of sites. This means that the operator B should have non-trivial components in an N-dimensional subspace of the space l^2, which makes the algebraic structure of the result more complicated, but does not lead to any essentially new spectral phenomena.

and eliminate the 'impurity channel' variable

$$f = -\beta\langle\Psi, \widehat{\chi}\rangle_{\mathcal{H}}\, R_d(z-\mu)\chi$$

where $R_d(z) = (H_d - z)^{-1}$. This leads to the effective equation

$$(H + W_{11}(z) - z)\,\Psi = 0 \tag{16}$$

in the space $\mathcal{H}$ with the energy-dependent interaction [17, 18]

$$W_{11}(z)* = -BR_d(z-\mu)B^+* = -\beta^2\langle *, \widehat{\chi}\rangle_{\mathcal{H}}\langle R_d(z-\mu)\chi, \chi\rangle_{l_2}\widehat{\chi}.$$

Using equation (16) one can write [17, 18] the Lippmann–Schwinger equation for the block $\widehat{R}_{11}(z) = (H + W_{11}(z) - z)^{-1}$ of the resolvent $(\widehat{H}_B - z)^{-1} = R_B(z) = \{\widehat{R}_{ij}\}_{i,j=1}^2$:

$$\widehat{R}_{11}(z) = R(z) - R(z)W_{11}(z)\widehat{R}_{11}(z). \tag{17}$$

This equation has an exact solution

$$\widehat{R}_{11}(z) = R(z) + \frac{\beta^2}{Q(z)}\langle R(z)*, \widehat{\chi}, \widehat{\chi}\rangle_{\mathcal{H}}\langle R_d(z-\mu)\chi, \chi\rangle_{l_2} \tag{18}$$

where the Krein determinant $Q(z)$ [17, 18] is given by

$$Q(z) = 1 - \beta^2\langle R(z)\widehat{\chi}, \widehat{\chi}\rangle_{\mathcal{H}}\langle R_d(z-\mu)\chi, \chi\rangle_{l_2}. \tag{19}$$

Similarly, one can eliminate from equation (15) the variable

$$\Psi(y) = -R(z)Bf = -\beta\langle f, \chi\rangle_{l_2} R(z)\widehat{\chi}(y).$$

In this case one obtains the effective equation

$$(H_d + \mu - z + W_{22}(z))\, f = 0$$

with the energy-dependent interaction

$$W_{22}(z)* = -B^+R(z)B = -\beta^2\langle *, \chi\rangle_{l_2}\langle R(z)\widehat{\chi}, \widehat{\chi}\rangle_{\mathcal{H}}\chi.$$

The correspondent Lippmann–Schwinger equation has again an exact solution

$$\widehat{R}_{22}(z) = R_d(z-\mu) + \frac{\beta^2}{Q(z)}\langle R(z)\widehat{\chi}, \widehat{\chi}\rangle_{\mathcal{H}}\langle R_d(z-\mu)*, \chi\rangle_{l^2}\, R_d(z-\mu)\chi. \tag{20}$$

The operators $\widehat{R}_{11}(z)$ and $\widehat{R}_{22}(z)$ are the diagonal elements of the resolvent of the operator $\widehat{H}_B$,

$$(\widehat{H}_B - z)^{-1} = \widehat{R}_B(z) = \begin{pmatrix} \widehat{R}_{11}(z) & \widehat{R}_{12}(z) \\ \widehat{R}_{21}(z) & \widehat{R}_{22}(z) \end{pmatrix}.$$

It remains to reconstruct the off-diagonal elements $\widehat{R}_{12}(z)$ and $\widehat{R}_{21}(z)$. To this end let us consider the Lippmann–Schwinger equation for the resolvent $\widehat{R}_B(z)$ of the operator $\widehat{H}_B$:

$$\widehat{R}_B(z) = \widehat{R}(z) - \widehat{R}(z)V\widehat{R}_B(z). \tag{21}$$

Here

$$\widehat{R}(z) = \begin{pmatrix} R(z) & 0 \\ 0 & R_d(z-\mu) \end{pmatrix}$$

is the resolvent of the operator $\widehat{H}$ and V is the perturbation. On the basis of the Lippman–Schwinger equation (21) the off-diagonal elements of the operator-valued matrix $\widehat{R}_B(z)$ are easily expressed through the diagonal ones as follows:

$$\widehat{R}_{12}(z) = -R(z)B\widehat{R}_{22}(z) \tag{22}$$

$$\widehat{R}_{21}(z) = -R_d(z-\mu)B^+\widehat{R}_{11}(z). \tag{23}$$

The analysis of the analytical structure of the constructed resolvent $\widehat{R}_B(z)$ leads to the following conclusions. The resolvent $\widehat{R}_B(z)$ is an analytic operator-valued function in the upper ($\mathrm{Im}\, z > 0$) and lower ($\mathrm{Im}\, z < 0$) half-planes. It has a jump on the continuous spectrum of the operator $\widehat{H}_B$ which coincides with the real axis $\mathbb{R}$. Such analytic properties are the direct consequence of the same analytic properties of the resolvent $R(z)$ and of the fact that the poles of the resolvent $R_d(z-\mu)$, namely the points $z_m = 2\pi\varepsilon a m + \mu$, $m \in \mathbb{Z}$, are cancelled in each matrix element of $\widehat{R}_B(z)$. This cancellation can be easily checked on use of the explicit formulae (18), (20), (22), and (23) for the matrix elements.

It should be noted that the resolvent $\widehat{R}(z)$ of the unperturbed operator $\widehat{H}$ besides the jump on the real axis also has poles at the points $z_m = \lambda_m + \mu$, $m \in \mathbb{Z}$. So the eigenvalues of the operator $\widehat{H}$ are embedded into the continuous spectrum. Thus we have shown that the adding of the impurity destroys these eigenvalues. In what follows we show that under the perturbation V eigenvalues convert into an infinite set of resonances.

We shall define resonances as the poles of the analytic continuation of the quadratic form (see [21] and references therein)

$$\tau(z) = \langle \widehat{R}_B(z)u, v\rangle_{\widehat{\mathcal{H}}}. \tag{24}$$

Here the vectors

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

are appropriate elements form the space $\widehat{\mathcal{H}} = \mathcal{H} \oplus l^2$ ($u_1, v_1 \in \mathcal{H}$ and $u_2, v_2 \in l^2$) to be specified later.

Let us denote as $\widehat{\mathcal{H}}_A$ the subset of the space $\widehat{\mathcal{H}}$ consisting of elements u whose components $u_1(y)$ admit analytic continuation into the strip $|\mathrm{Re}\, y| < \pi$. The space $\widehat{\mathcal{H}}_A$ is dense in $\widehat{\mathcal{H}}$. Indeed, since $u_1(y) \in L_2(-\pi,\pi) \times l^2$, one can take polynomials as a dense subset in $L_2(-\pi,\pi)$ whose elements admit the above analytic continuation. Let us introduce the notation $\lambda_m^{\pm} = \lambda_m \pm \pi\varepsilon a$. The following statement is valid.

Lemma 2. Let $u, v \in \widehat{\mathcal{H}}_A$. Then the form $\tau(z)$ admits meromorphic continuation from above (below) to below (above) in any strip $S_m = \{z : \lambda_m^- < \mathrm{Re}\, z < \lambda_m^+\}$ and its poles coincide with the zeros of the correspondent continuation of the Krein's determinant $Q(z)$.

Proof. Let us rewrite the quadratic form (24) as follows:

$$\tau(z) = \langle R_{11}(z)u_1 + R_{12}(z)u_2, v_1\rangle_{\mathcal{H}} + \langle R_{21}(z)u_1 + R_{22}(z)u_2, v_2\rangle_{l^2}. \tag{25}$$

Since the proof for all terms in equation (25) is similar, we show only that $\langle R_{11}(z)u_1, v_1\rangle_{\mathcal{H}}$ admits analytic continuation into the strip $\lambda_m^- < \mathrm{Re}\, z < \lambda_m^+$. Using equation (18) one obtains

$$\langle R_{11}(z)u_1, v_1\rangle_{\mathcal{H}} = \langle R(z)u_1, v_1\rangle_{\mathcal{H}} + \frac{\beta^2}{Q(z)}\langle R_d(z-\mu)\chi, \chi\rangle_{l^2}\langle R(z)u_1, \widehat{\chi}\rangle_{\mathcal{H}}\langle R(z)\widehat{\chi}, v_1\rangle_{\mathcal{H}}. \tag{26}$$

First let us show that the first term on the right-hand side of equation (26) admits analytic continuation into the strip S_m for any m. On use of equation (9) we have

$$\langle R(z)u_1, v_1\rangle_{\mathcal{H}} = \sum_{mnn'} J_{n-m}(\Theta)J_{n'-m}(\Theta)\varphi_m^{nn'}(z) \tag{27}$$

where

$$\varphi_m^{nn'}(z) \stackrel{\text{def}}{=} \int_{-\pi}^{\pi} \frac{(u_1(y))_{n'}\,(v_1(y))_n^*}{\lambda_m + \varepsilon a y - z}\,\mathrm{d}y. \tag{28}$$

Since the function $\varphi_m^{nn'}(z)$ is given by the Cauchy-type integral, it is analytic on the complex plane of variable z except the interval $[\lambda_m^-, \lambda_m^+]$. Let us introduce the following functions on the strip $\lambda_m^- < |\mathrm{Re}\, z| < \lambda_m^+$:

$$^{-}\varphi_m^{nn'}(z) \stackrel{\text{def}}{=} \begin{cases} \varphi_m^{nn'}(z) & \mathrm{Im}\, z > 0 \\ \varphi_m^{nn'}(z) + \dfrac{2\pi\mathrm{i}}{\varepsilon a} h_{nn'}\left(\dfrac{z-\lambda_m}{\varepsilon a}\right) & \mathrm{Im}\, z < 0 \end{cases} \tag{29}$$

and

$$^{+}\varphi_m^{nn'}(z) \stackrel{\text{def}}{=} \begin{cases} \varphi_m^{nn'}(z) - \dfrac{2\pi\mathrm{i}}{\varepsilon a} h_{nn'}\left(\dfrac{z-\lambda_m}{\varepsilon a}\right) & \mathrm{Im}\, z > 0 \\ \varphi_m^{nn'}(z) & \mathrm{Im}\, z < 0 \end{cases} \tag{30}$$

where

$$h_{nn'}(z) \stackrel{\text{def}}{=} (u_1(z))_{n'}\,(v_1(z))_n^*$$

is the analytic continuation of the Cauchy-type integral density (28) into the strip S_m.

One can check that the functions $^{\pm}\varphi_m^{nn'}(z)$ are analytic in the strip S_m. To this end it is enough to show that

$$^{\pm}\varphi_m^{nn'}(\lambda + \mathrm{i}0) = {}^{\pm}\varphi_m^{nn'}(\lambda - \mathrm{i}0)$$

for any $\lambda \in \mathbb{R}$. This relation follows directly from the limit values of the Cauchy-type integral $\varphi_m^{nn'}(z)$ on the interval $(\lambda_m^-, \lambda_m^+)$,

$$\varphi_m^{nn'}(\lambda \pm \mathrm{i}0) = \pm\frac{\pi\mathrm{i}}{\varepsilon a} h_{nn'}\left(\frac{\lambda-\lambda_m}{\varepsilon a}\right) + \mathrm{PV}\int_{-\pi}^{\pi} \frac{h_{nn'}(y)}{\lambda_m + \varepsilon a y - \lambda}\,\mathrm{d}y. \tag{31}$$

Thus the function $^{-}\varphi_m^{nn'}(z)$ is the analytic continuation of the function $\varphi_m^{nn'}(z)$ from above to below and the function $^{+}\varphi_m^{nn'}(z)$ is the analytic continuation of the function $\varphi_m^{nn'}(z)$ from below to above in the strip S_m. Hence the form $\langle R(z)u_1, v_1\rangle_{\mathcal{H}}$ given by the series (27) admits analytic continuation from above to below and *vice versa* in each strip S_m. From equations (29)–(31) it follows that these continuations through the interval $(\lambda_m^-, \lambda_m^+)$ have the form

$$\langle R(z)u_1, v_1\rangle_{\mathcal{H}}^{\mp} = \sum_{lnn'} J_{n-l}(\Theta) J_{n'-l}(\Theta) \varphi_l^{nn'}(z) \pm \frac{2\pi\mathrm{i}}{\varepsilon a} \sum_{nn'} J_{n-m}(\Theta) J_{n'-m}(\Theta) h_{nn'}\left(\frac{z-\lambda_m}{\varepsilon a}\right). \tag{32}$$

These functions are analytic in the strip S_m. They can also be considered as the fixed branches of the functions given by the same formulae (27) and (28) on the complex plane of variable z with two cuts along the rays $(-\infty, \lambda_m^-]$ and $[\lambda_m^+, \infty)$.

Let us consider now the second term in the right-hand side of equation (26). The analytic continuation of the factors $\langle R(z)u_1, \widehat{\chi}\rangle_{\mathcal{H}}$ and $\langle R(z)\widehat{\chi}, v_1\rangle_{\mathcal{H}}$ is a consequence of the analytic continuation proven above for the form $\langle R(z)u_1, v_1\rangle_{\mathcal{H}}$. The factor $\langle R_d(z-\mu)\chi, \chi\rangle_{l^2}$ is a meromorphic function with the poles at the points $\lambda_m + \mu$. However, one can check that these poles are cancelled by the same poles of the Krein determinant $Q(z)$ (and its analytic continuations $Q^{\pm}(z)$ as well).

The last factor to be considered is the Krein determinant $Q(z)$. Using equations (9), (19), (27) and (28) one can rewrite it as

$$Q(z) = 1 - \frac{\beta^2}{\varepsilon a}\left(\sum_{p=-\infty}^{\infty}\frac{J_p^2(\Theta)}{\lambda_p - z + \mu}\right)\left(\sum_{p=-\infty}^{\infty}J_p^2(\Theta)\ln\left(\frac{\lambda_p^+ - z}{\lambda_p^- - z}\right)\right). \qquad (33)$$

Here the branch of the logarithm is chosen such that

$$\ln\left(\frac{\lambda_p^+ - z}{\lambda_p^- - z}\right)\bigg|_{z=\lambda+\mathrm{i}0} = \mathrm{i}\pi + \ln\left(\frac{\lambda_p^+ - \lambda}{\lambda - \lambda_p^-}\right) \qquad \lambda \in (\lambda_p^-, \lambda_p^+). \qquad (34)$$

One can check that the analytic continuation of the function $Q(z)$ into the strip S_m from above (below) to below (above) is given by the function $Q^-(z)$ $(Q^+(z))$:

$$Q_m^-(z) \stackrel{\text{def}}{=} \begin{cases} Q(z) & \operatorname{Im} z > 0 \\ Q(z) - \dfrac{2\pi\mathrm{i}\beta^2}{\varepsilon a}J_m^2(\Theta)\displaystyle\sum_{p=-\infty}^{\infty}\frac{J_p^2(\Theta)}{\lambda_p - z + \mu} & \operatorname{Im} z < 0 \end{cases} \qquad (35)$$

$$Q_m^+(z) \stackrel{\text{def}}{=} \begin{cases} Q(z) + \dfrac{2\pi\mathrm{i}\beta^2}{\varepsilon a}J_m^2(\Theta)\displaystyle\sum_{p=-\infty}^{\infty}\frac{J_p^2(\Theta)}{\lambda_p - z + \mu} & \operatorname{Im} z > 0 \\ Q(z) & \operatorname{Im} z < 0. \end{cases} \qquad (36)$$

Therefore the form $\langle R_{11}(z)u_1, v_1\rangle_{\mathcal{H}}$ can be continued in each strip S_m as a meromorphic function and the only poles can be given by the zeros of the Krein determinant continuations $Q^{\pm}(z)$.

The proof of the statement of lemma 2 for all other terms of the form $\tau(z)$ is similar. This accomplishes the proof of the lemma. □

From the above proof one can see that if the functions $Q_m^-(z)$ and $Q_m^+(z)$ have zeros z_m^- and z_m^+ in the half-strips $S_m^- = S_m \cap \{z : \operatorname{Im} z < 0\}$ and $S_m^+ = S_m \cap \{z : \operatorname{Im} z > 0\}$, respectively. These points are resonances.

To analyse the localization of the resonances let us consider the weak coupling limit $\beta \ll 1$.

The following statement is valid.

Theorem 1. The quadratic form $\tau(z) = \langle \widehat{R}_B(z)u, v\rangle_{\widehat{\mathcal{H}}}$ for any $u, v \in \widehat{\mathcal{H}}$ is an analytic function in upper ($\operatorname{Im} z > 0$) and lower ($\operatorname{Im} z < 0$) half-planes and has a jump on the real axis. The meromorphic continuations $\tau_m^{\pm}(z)$ of $\tau(z)$ into the strip S_m have a set of poles (resonances) in the upper and lower half-strips, respectively, which coincide with zeros of continuations $Q_m^{\pm}(z)$ of the Krein determinant $Q(z)$. In the weak-coupling limit ($\beta \ll 1$) there is at least one pole of $\tau_m^+(z)$ $(\tau_m^-(z))$ in the lower (upper) half-strip for every $m \in \mathbb{Z}$ given by

$$z_m^{\pm} \underset{\beta\ll 1}{=} \lambda_{M(\lambda_m-\mu)} + \mu - \frac{\beta^2}{\varepsilon a}J_{M(\lambda_m-\mu)}^2(\Theta)\sum_{p=-\infty}^{\infty}J_p^2(\Theta)\ln\left|\frac{\lambda_p^+ - \lambda_{M(\lambda_m-\mu)} - \mu}{\lambda_p^- - \lambda_{M(\lambda_m-\mu)} - \mu}\right|$$
$$-3\mathrm{i}\pi\frac{\beta^2}{\varepsilon a}J_{M(\lambda_m-\mu)}^2(\Theta)J_m^2(\Theta) + \mathrm{O}(\beta^4). \qquad (37)$$

Here the Bessel function subindex $M(\lambda_m - \mu)$ is given by the integer-valued function $M(\lambda) = [(\lambda/2\pi\varepsilon a) + (1/2)]$ and $\lambda_m^{\pm} = \pi\varepsilon a(2m \pm 1)$.

Proof. Analyticity of the function $\tau(z)$ in $\mathbb{C}\backslash\mathbb{R}$ is a straightforward consequence of the analytic properties of the resolvent $\widehat{R}(z)$ of the self-adjoint operator $\widehat{H}_B$.

Lemma 2 implies that singularities of the continuations $\tau_m^{\pm}(z)$ coincide with the zeros of the continuations $Q_m^{\pm}(z)$ of Krein's determinant $Q(z)$. In the upper half-strip $Q_m^{+}(z) = Q(z)$ and has no zeros, while in the lower half-strip S_m^- due to equations (33) and (35) zeros of $Q_m^+(z)$ are given by the roots of the equation

$$\frac{\beta^2}{\varepsilon a}\sum_{n=-\infty}^{\infty}\frac{J_n^2(\Theta)}{\lambda_n - z + \mu}\left(\sum_{p=-\infty}^{\infty}J_p^2(\Theta)\ln\left(\frac{\lambda_p^+ - z}{\lambda_p^- - z}\right) + 2\mathrm{i}\pi J_m^2(\Theta)\right) = 1. \tag{38}$$

Let us show that in the weak coupling limit $\beta \ll 1$ at least one pole of the resolvent $(H_d - z + \mu)^{-1}$ generates a root of equation (38) and, consequently, a pole of $\tau_m^+(z)$ in the lower half-plane (resonance). Considering $\tau_m^-(z)$ one should just replace the lower half-plane by the upper one and *vice versa*.

We assume $\mu \neq n + 1/2$, $n \in \mathbb{Z}$, and choose the index $m' \in \mathbb{Z}$ such that $\lambda_{m'} + \mu \in (\lambda_m^-, \lambda_m^+)$. This means

$$m' = M(\lambda_m - \mu). \tag{39}$$

Multiplying both sides of equation (38) by the factor $(\lambda_{m'} - z - \mu)$ and using the expansion in powers of β^2 near $\beta = 0$ we find the root

$$z_m^+ = \lambda_{m'} + \mu - \frac{\beta^2}{\varepsilon a}J_{m'}^2(\Theta)\left(\sum_{p=-\infty}^{\infty}J_p^2(\Theta)\,\ln\left(\frac{\lambda_p^+ - \lambda_{m'} - \mu}{\lambda_p^- - \lambda_{m'} - \mu}\right) + 2\mathrm{i}\pi J_m^2(\Theta)\right). \tag{40}$$

The argument of the logarithm on the right side of equation (40) is negative iff $\lambda_p^- - \mu < \lambda_{m'} < \lambda_p^+ - \mu$, i.e. when $p = M(\lambda_{s(m)} + \mu)$. One use of equation (39) is that we find that it implies $p = m$. Finally, using equations (34), (39), and (40) we get the resonance in the lower half-plane given by equation (37). Calculations of the resonances z_m^- in the upper half-plane are obviously similar. The theorem is proved. □

4. Lee–Friedrichs representation

The Hamiltonian $\widehat{H}$ acts in the Hilbert space $\widehat{\mathcal{H}}$ which can be represented as the orthogonal sum $\widehat{\mathcal{H}} = l^2(\mathbb{Z}, L^2[-\pi,\pi)) \oplus l^2$. In accordance with this representation let us introduce the generalized Friedrichs states $|\omega\rangle$ and $|s\rangle$ [16, 19, 22–24] as follows,

$$|\omega\rangle = \begin{pmatrix} |\tilde{\omega}\rangle \\ 0 \end{pmatrix}$$

where $|\tilde{\omega}\rangle = \widehat{J}^{(M(\omega))}\delta(y + 2\pi M(\omega) - (\omega/\varepsilon a)$

$$|s\rangle = \begin{pmatrix} 0 \\ |\tilde{s}\rangle \end{pmatrix}$$

and $|\tilde{s}\rangle = \widehat{J}^{(s)} = \{J_{n-s}\}_{n\in\mathbb{Z}} \in l^2$. Then one can check that the perturbed Hamiltonian $\widehat{H_B}$ can be written in the Lee–Friedrichs representation

$$\widehat{H_B} = 2\pi a\varepsilon\sum_{s=-\infty}^{\infty}s|s\rangle\langle s| + \int_{-\infty}^{\infty}\mathrm{d}\omega\,\omega|\omega\rangle\langle\omega| + \sum_{s=-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\omega\,v_s(\omega)\,(|\omega\rangle\langle s| + |s\rangle\langle\omega|) \tag{41}$$

where the spectral density $v_s(\omega)$ reads as

$$v_s(\omega) = \beta J_{-M(\omega)}(\Theta)J_{-s}(\Theta) \qquad \omega \in \mathbb{R}.$$

The representation (41) shows that our model can be effectively studied using methods developed in various papers [16, 22, 23, 25–27] for the Friedrichs model. The main interest in this model is related to consideration of resonance states associated with resonance poles of the extended resolvent, their interpretation as decaying states and construction of generalized spectral decomposition [19, 22–24, 28]. However, one should note that the above constructed spectral density $v_s(\omega)$ cannot be analytically continued from the real axis neither to the upper nor to the lower half-plane, thus we cannot directly use the technique of generalized spectral decomposition elaborated in [28] for the Friedrichs model to the discrete Stark Hamiltonian under consideration.

In conclusion let us note that the natural continuation of the present study would be the construction of a generalized spectral decomposition for the extended discrete Stark Hamiltonian in a rigged Hilbert space $\Phi_\pm \subset \widehat{\mathcal{H}} \subset \Phi_\pm^\dagger$ and the proof of the weak completeness here. On this basis one can split the unitary evolution group into two semigroups and study the problem of intrinsic irreversibility. However, it seems that the solution of these problems should be the subject of a separate paper.

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